Q→ Prove that (i) $div(\vec{A}\times\vec{B}) = \vec{B}\cdot(curl\,\vec{A}) - \vec{A}\cdot(curl\,\vec{B})$.

Proof →

$$Div(\vec{A}\times\vec{B}) = \nabla\cdot(\vec{A}\times\vec{B})$$

$$= \sum i\frac{\partial}{\partial x}(\vec{A}\times\vec{B})$$

$$= \sum \hat{i}\left(\frac{\partial\vec{A}}{\partial x}\times\vec{B} + \vec{A}\times\frac{\partial\vec{B}}{\partial x}\right)$$

$$= \sum \hat{i}\,\frac{\partial\vec{A}}{\partial x}\times\vec{B} + \sum\hat{i}\cdot\vec{A}\times\frac{\partial\vec{B}}{\partial x}$$

$$= \sum\hat{i}\,\frac{\partial\vec{A}}{\partial x}\times\vec{B} - \sum\hat{i}\cdot\frac{\partial B}{\partial x}\times\vec{A} \quad [\because \vec{A}\times\vec{B} = -\vec{B}\times\vec{A}]$$

$$= \sum\hat{i}\times\frac{\partial\vec{A}}{\partial x}\times\vec{B} - \sum\hat{i}\times\frac{\partial\vec{B}}{\partial x}\cdot\vec{A}$$

$$= \sum \int\vec{i}\times\frac{\partial\vec{A}}{\partial x}\int\vec{B} - \sum\int\vec{i}\times\frac{\partial B}{\partial x}\int\vec{A}$$

$$= (\nabla\times\vec{A})\cdot\vec{B} - (\nabla\times\vec{B})\vec{A}$$

$$= \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B})$$

$$= \vec{B}\cdot(curl\,\vec{A}) - \vec{A}\cdot(curl\,\vec{B})$$ Proved

2Q → Prove that $curl\,curl\,\vec{A} = grad\,div A - \nabla^2\vec{A}$.

Proof →

$$curl\,curl\,\vec{A} = \nabla\times(\nabla\times\vec{A})$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\times\left(\frac{\partial A}{\partial x}\vec{i} + \frac{\partial\vec{A}}{\partial y}\hat{j} + \frac{\partial\vec{A}}{\partial z}\vec{k}\right)$$

$$= \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\times\left[\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{i} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{k}\right]$$

$$= \sum \hat{i}\left[\hat{i} \times \frac{\partial^2 \vec{A}}{\partial x^2} + \hat{j} \times \frac{\partial^2 \vec{A}}{\partial x \partial y} + \hat{k} \times \frac{\partial^2 \vec{A}}{\partial x \partial z}\right]$$

$$= \sum \left[\vec{i} \times \left(\hat{i} \times \frac{\partial^2 \vec{A}}{\partial x^2}\right) + \hat{i} \times \left(\hat{j} \times \frac{\partial^2 \vec{A}}{\partial x \partial y}\right) + \hat{i} \times \left(\hat{k} \times \frac{\partial^2 \vec{A}}{\partial x \partial z}\right)\right]$$

$$= \sum \hat{i} \cdot \frac{\partial^2 A}{\partial x^2}\hat{i} - (\hat{i}\cdot\hat{i})\frac{\partial^2 \hat{A}}{\partial x^2} + \left(\hat{i}\frac{\partial^2 A}{\partial x^2}\right)\hat{j} - (\hat{i}\cdot\hat{i})\frac{\partial^2 A}{\partial x^2}$$

$$+ \left(\hat{i}\cdot\frac{\partial^2 \vec{A}}{\partial x \partial z}\right)\hat{k} - (\hat{i}\cdot\hat{k})\frac{\partial^2 \vec{A}}{\partial x \partial z}\Big]$$

$$= \left[\left(\hat{i}\cdot\frac{\partial^2 \vec{A}}{\partial x^2}\right)\hat{i} + \left(\hat{i}\cdot\frac{\partial^2 \vec{A}}{\partial x \partial y}\right)\hat{j} + \left(\hat{i}\cdot\frac{\partial^2 \vec{A}}{\partial x \partial z}\right)\hat{k}\right] - \sum \frac{\partial^2 \vec{A}}{\partial x^2} \quad \text{(i)}$$

But by definition, we know that

$$\nabla\cdot\vec{A} = \hat{i}\frac{\partial \vec{A}}{\partial x} + \hat{j}\frac{\partial \vec{A}}{\partial y} + \vec{k}\cdot\frac{\partial A}{\partial z}$$

$$\therefore \nabla\cdot(\nabla\cdot\vec{A}) = \sum \hat{i}\frac{\partial}{\partial x}\left(\hat{i}\frac{\partial \vec{A}}{\partial x} + \hat{j}\frac{\partial \vec{A}}{\partial y} + k\frac{\partial A}{\partial z}\right)$$

or, $$\nabla\cdot(\nabla\cdot\vec{A}) = \sum\left[\left(\hat{i} \times \frac{\partial^2 A}{\partial x^2}\right)\hat{i} + \left(\hat{i} \times \frac{\partial^2 A}{\partial x \partial y}\right)\hat{j} + \left(\hat{k} \times \frac{\partial^2 A}{\partial x \partial y}\right)\hat{k}\right]$$

Substituting this value in (i), we get

$$\nabla \times (\nabla \times \vec{A}) = \nabla\cdot(\nabla\cdot\vec{A}) - \sum \frac{\partial^2 A}{\partial x^2}$$

or, curl curl $\vec{A}$ = grad div $\vec{A} - \nabla^2\vec{A}$. Proved

Q:→ Prove that curl(grad ϕ) = 0

or, $\nabla \times (\nabla\phi) = 0$.

Proof → L.H.S = curl(grad ϕ) = $\nabla \times (\nabla\phi)$

$$= \nabla \times \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial y \partial z}\right) + \hat{j}\left(\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial x \partial z}\right) + \hat{k}\left(\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial x \partial y}\right)$$

$= 0$ R.H.S. Proved.

Q Prove that Div curl $\vec{A} = 0$

Proof → L.H.S. $= \text{Div}(\text{curl}\,\vec{A}) = \nabla\cdot(\nabla \times \vec{A})$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\sum \hat{i} \times \frac{\partial V_x}{\partial x}\right)$$

$$= \nabla\cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left[\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\hat{i} + \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)\hat{j} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\hat{k}\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial V_z}{\partial z} - \frac{\partial V_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

$$= \cancel{\frac{\partial^2 V_z}{\partial x \partial y}} - \cancel{\frac{\partial^2 V_y}{\partial x \partial y}} + \cancel{\frac{\partial^2 V_x}{\partial y \partial z}} - \cancel{\frac{\partial^2 V_z}{\partial x \partial y}} + \cancel{\frac{\partial^2 V_y}{\partial x \partial z}} - \cancel{\frac{\partial^2 V_x}{\partial y \partial z}} = 0$$

R.H.S.
Proved

Q→ Prove that div (u grad v) = $u \nabla^2 v$ + (grad u) (grad v)
where 'u' and 'v' both are scalar point function.

Proof:→ Let us put $\vec{V}$ = grad v

$$\text{div}(u \text{ grad } v) = \text{div}(u\vec{v})$$
$$= u \text{ div } V + V \cdot \text{grad } u$$
$$= u \text{ div}(\text{grad } v) + (\text{grad } u)\cdot(\text{grad } u)$$
$$= u \nabla\cdot\nabla u + (\text{grad } u)\cdot(\text{grad } u)$$
$$= u\nabla^2 u + (\text{grad } u)(\text{grad } v)$$
$$= u\nabla^2 V + \nabla u \cdot \nabla V$$

or, $\text{div}(u \text{ grad } v) = u\nabla^2 V + \nabla u \cdot \nabla u$

or, $\text{div}(u \text{ grad } v) = u\nabla^2 v + (\text{grad } u)\cdot(\text{grad } v)$

Proved

Q→ If $\vec{A}$ is a constant vector, prove that $\nabla(\vec{r}\cdot\vec{A}) = \vec{A}$
where $\vec{r}$ is a position vector.

Proof→ $\because \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

and $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

$$\therefore \vec{r}\cdot\vec{A} = (x\hat{i} + y\hat{j} + z\hat{k})\cdot(A_x\hat{i} + A_y\hat{j} + A_z\hat{k})$$
$$= xA_x + yA_y + zA_z$$
$$\therefore \nabla(\vec{r}\cdot\vec{A}) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot(xA_x + yA_y + zA_z)$$
$$= \frac{\partial x}{\partial x}A_x\hat{i} + \frac{\partial y}{\partial y}A_y\hat{j} + \frac{\partial z}{\partial z}A_z\hat{k}$$
$$= A_z\hat{i} + A_y\hat{j} + A_z\hat{k}$$
$$= \vec{A}$$ R.H.S. Proved